2018-2019 MM2MSD Exam Solutions

SECTION A

1.

D. 135.5 MPa

[2 marks]

SOLUTION 1

Given $\sigma_z = 125\text{ MPa}$, $\sigma_y = 50\text{ MPa}$ and $\tau_{zy} = 30\text{ MPa}$

$$C = \frac{\sigma_z + \sigma_y}{2} = \frac{125 + 50}{2} = 87.5\text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_z - \sigma_y}{2}\right)^2 + {\tau_{zy}}^2} = \sqrt{\left(\frac{125 - 50}{2}\right)^2 + 30^2} = 48.0\text{ MPa}$$

$$\sigma_1 = C + R = 87.5 + 48 = \mathbf{135.5\ MPa}$$

2.

A. Yielding and buckling, respectively

[2 marks]

3.

E. Linear hardening

[2 marks]

4.

B. 37.5 MPa

[2 marks]

SOLUTION 4

For a rectangular cross-section

$$I = \frac{bd^3}{12} = \frac{20 \times 40^3}{12} = 106666.7 \text{ mm}^4$$

The shear stress in a rectangular cross section is given by

$$\tau = \frac{SA\bar{y}}{Iz}$$

The max shear stress value occurs at the N.A., therefore $\bar{y} = \frac{40}{4} = 10 \text{ mm}$ and $A = 20 \times 20 = 400 \text{ mm}^2$, the width of the section, $z = 20 \text{ mm}$

$$\tau = \frac{SA\bar{y}}{Iz} = \frac{20000 \times 400 \times 10}{106666.7 \times 20} = \mathbf{37.5\ MPa}$$

Or, for a rectangular cross section:

$$\tau = 1.5\frac{S}{bd} = 1.5 \times \left(\frac{20000}{20 \times 40}\right) = \mathbf{37.5\ MPa}$$

5.

E. $A = \frac{15625}{1875}, B = \frac{15625}{3}$

[2 marks]

SOLUTION 5

Lame's equations:

$$\sigma_r = A - \frac{B}{r^2}$$

$$\sigma_\theta = A + \frac{B}{r^2}$$

BCs,

at $r = 12.5 \text{ mm}$, $\sigma_r = -25 \text{ MPa}$

$$-25 = A - \frac{B}{156.25}$$

at $r = 25 \text{ mm}$, $\sigma_r = 0 \text{ MPa}$

$$0 = A - \frac{B}{625}$$

Giving

$$A = \frac{B}{625}$$

and,

$$-25 = B\left(\frac{1}{625} - \frac{1}{156.25}\right)$$

$$-25 = B\left(\frac{1-4}{625}\right)$$

Therefore,

$$B = -25 \times \left(\frac{625}{-3}\right) = \mathbf{\frac{15625}{3}} \;\; (5208.2)$$

and,

$$A = \frac{B}{625} = \frac{\mathbf{15625}}{\mathbf{1875}}\ (8.333)$$

6.

C. $u = \frac{2P}{EI}\left(\frac{L^3}{6} + \frac{3\pi L^2 R}{2} + \frac{4\pi R^3}{5} + \pi R^2\right)$

[2 marks]

SOLUTION 6

Deflection, u, at the position of and in the direction of load, P, is:

$$u = \frac{\partial U}{\partial P}$$

$$\therefore \boldsymbol{u} = \frac{\mathbf{2}\boldsymbol{P}}{\boldsymbol{EI}}\left(\frac{\boldsymbol{L}^{\mathbf{3}}}{\mathbf{6}} + \frac{\mathbf{3}\boldsymbol{\pi}\boldsymbol{L}^{\mathbf{2}}\boldsymbol{R}}{\mathbf{2}} + \frac{\mathbf{4}\boldsymbol{\pi}\boldsymbol{R}^{\mathbf{3}}}{\mathbf{5}} + \boldsymbol{\pi}\boldsymbol{R}^{\mathbf{2}}\right)$$

7.

A. OA

[2 marks]

8.

E. No

[2 marks]

SOLUTION 8

Behaviour is assumed to be all elastic and therefore:

$$\frac{M_y}{I} = \frac{\sigma_y}{y}$$

where M_y is the moment required to cause yielding.

First yield will occur at $y = \pm\frac{d}{2}$, i.e. at the top and bottom edges:

$$\therefore M_y = \frac{\sigma_y \times I}{y} = \frac{\sigma_y \times \left(\frac{bd^3}{12}\right)}{\frac{d}{2}} = \frac{215 \times \left(\frac{125 \times 225^3}{12}\right)}{\frac{225}{2}} = 226{,}757{,}812.5Nmm = 226.76kNm$$

Since $M < M_y$, **yielding does not occur**.

9.

E. $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

[2 marks]

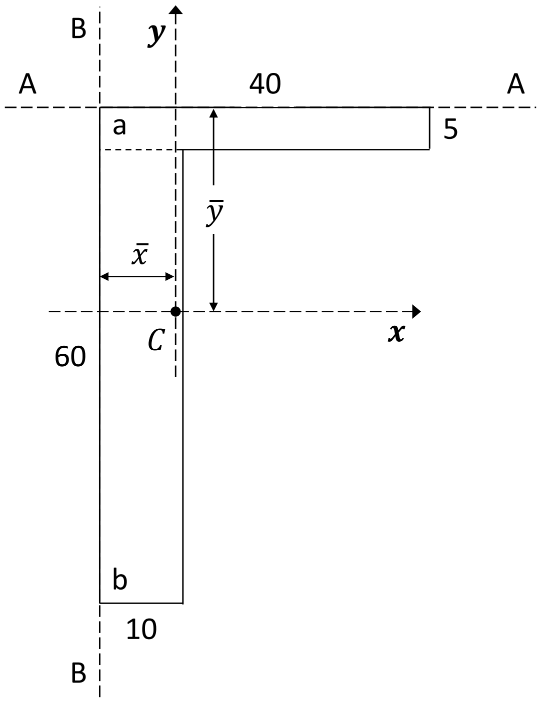
10.

E. $\bar{x} = 9\ mm, \bar{y} = 24.5\ mm$

[2 marks]

SOLUTION 10

Splitting the cross section into two rectangles (a and b) and taking the top and left edges to be the datum edges (AA and BB, respectively):



Total area,

$$A = (40 \times 5)_{\mathrm{a}} + (55 \times 10)_{\mathrm{b}} = 750\ \mathrm{mm}^4$$

Taking moments about AA:

$$A\bar{y} = A_a y_a + A_b y_b$$

$$\therefore \bar{y} = \frac{(40 \times 5 \times 2.5)_{\mathrm{a}} + (10 \times 55 \times 32.5)_{\mathrm{b}}}{750} = \mathbf{24.5\ mm}$$

Similarly, taking moments about BB:

$$A\bar{x} = A_{\mathrm{a}} x_{\mathrm{a}} + A_{\mathrm{b}} x_{\mathrm{b}}$$

$$\therefore \bar{x} = \frac{(5 \times 40 \times 20)_\mathrm{a} + (55 \times 10 \times 5)_\mathrm{b}}{750} = \mathbf{9\ mm}$$

11.

C. Less conservative than the Goodman line

[2 marks]

12.

C. C

[2 marks]

13.

C. 43 MPa

[2 marks]

SOLUTION 13

Axial stress

$$\sigma_a = \frac{F}{A} = \frac{40000}{\pi \times 20^2} = 32\ \mathrm{MPa}$$

Torsional shear stress

$$\tau = \frac{Tr}{J} = \frac{2 \times 500}{\pi r^3} = 40\ \mathrm{MPa}$$

Maximum shear stress is radius of Mohrs circle, given by:

$$R = \tau_{max} = \sqrt{\left(\frac{\sigma_a}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{32}{2}\right)^2 + 40^2} = \mathbf{43\ MPa}$$

14.

A. 46 MPa

[2 marks]

SOLUTION 14

$$\sigma_r = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2$$

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

at $r = 0.05$ (ID), $\sigma_r = 0$ therefore:

$$0 = A - \frac{B}{0.05^2} - \frac{3+0.3}{8}7900 \times 209.4^2 \times 0.05^2$$

$$0 = A - 400B - 3.57 \times 10^5$$

at $r = 0.4$ (OD), $\sigma_r = 0$ therefore:

$$0 = A - \frac{B}{0.4^2} - \frac{3+0.3}{8}7900 \times 209.4^2 \times 0.4^2$$

$$0 = A - 6.25B - 22.9 \times 10^6$$

$$0 = -393.75B + 22.5 \times 10^6$$

therefore:

$$B = 57142$$

$$A = 23.2 \times 10^6$$

Hoop stress at the bore is given by:

$$\sigma_\theta = 23.2 \times 10^6 + \frac{57142}{0.05^2} - \frac{1+3\nu}{8}7900 \times 209.4^2 \times 0.05^2 = \mathbf{45.9\ MPa} \approx \mathbf{46\ MPa}$$

15.

B. 1.44 MPa

[2 marks]

SOLUTION 15

vM yield criterion for 2D plane stress:

$$\sigma_y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2$$

For an internally pressurised cylinder:

$$\sigma_1 = \sigma_\theta = \frac{pr}{t}$$

$$\sigma_2 = \sigma_r = \frac{pr}{2t}$$

Therefore:

$$\sigma_y^2 = \left(\frac{pr}{t}\right)^2 + \left(\frac{pr}{2t}\right)^2 - \left(\frac{pr}{t}\right)\left(\frac{pr}{2t}\right)$$

$$\sigma_y^2 = \frac{p^2r^2}{t^2} + \frac{p^2r^2}{4t^2} - \frac{p^2r^2}{2t^2} = \frac{3(p^2r^2)}{4t^2}$$

$$p = \sqrt{\frac{4\sigma_y^2 t^2}{3r^2}} = \sqrt{\frac{4 \times 250^2 \times 5^2}{3 \times 1000^2}} = \mathbf{1.44\ MPa}$$

16.

D. kinematic hardening

[2 marks]

17.

C. $\frac{dy}{dx} = \frac{1}{EI}\left(R_A \frac{x^2}{2} + M_O\langle x-3\rangle - w\frac{\langle x-6\rangle^3}{6} + A\right)$

[2 marks]

SOLUTION 17

Integrating $EI\frac{d^2y}{dx^2} = R_A x + M_O\langle x-3\rangle^0 - w\frac{\langle x-6\rangle^2}{2}$ with respect to x gives:

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} + M_O\langle x-3\rangle - w\frac{\langle x-6\rangle^3}{6} + A$$

Rearranging this for $\frac{dy}{dx}$:

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{EI}\left(R_A\frac{x^2}{2} + M_O\langle x-3\rangle - w\frac{\langle x-6\rangle^3}{6} + A\right)}$$

18.

D. 77 MPa

[2 marks]

SOLUTION 18

$$\delta_{total} = \delta_{mech} + \delta_{therm}$$

$$\delta_{total} = 0 = \frac{FL}{AE} + L\alpha\Delta T$$

Therefore:

$$\frac{F}{A} = -\alpha\Delta TE = -22\times10^{-6}\times -50\times 70\times10^{9} = 77\times10^{6}\text{ Pa} = \mathbf{77\ MPa}$$

19.

E. $P_c = \frac{4\pi^2 EI}{L^2}$

[2 marks]

20.

B. $U = \int_0^L \frac{M^2}{2EI} ds$

[2 marks]

SECTION B

21.

(a)

$$c_0 = \pi d_0 = \pi \times 20 = 62.8 \text{ mm}$$

$$c_1 = \pi d_1$$

$$c_1 - c_0 = \delta c = \pi \times 0.02 = 0.063 \text{ mm}$$

$$\delta c = c_0 \alpha \Delta T_{min}$$

$$\Delta T_{min} = \frac{\delta c}{c_0 \alpha} = \frac{0.063}{62.8 \times 12 \times 10^{-6}} = \mathbf{83.4\ °C}$$

[5 marks]

(b)

Applying Lame's equations for the inner cylinder (1):

$$\sigma_{r1} = A_1 - \frac{B_1}{r^2}$$

$$\sigma_{\theta 1} = A_1 + \frac{B_1}{r^2}$$

at $r = 5 \text{ mm}, \sigma_r = 0$

$$0 = A_1 - \frac{B_1}{25}$$

$$B_1 = 25A_1 \qquad (1)$$

at $r = 10 \text{ mm}, \sigma_r = -p$ (interface pressure)

$$-p = A_1 - \frac{B_1}{100}$$

If we substitute equation (1) into this, we can remove B_1 from the expression:

$$-p = A_1\left(1 - \frac{25}{100}\right)$$

And rearrange to give:

$$A_1 = -\frac{4}{3}p$$

[2 marks]

Therefore:

$$B_1 = 25 \times -\frac{4}{3}p$$

$$B_1 = -\frac{100}{3}p$$

[2 marks]

Lame's equations for the outer cylinder (2):

$$\sigma_{r2} = A_2 - \frac{B_2}{r^2}$$

$$\sigma_{\theta 2} = A_2 + \frac{B_2}{r^2}$$

At $r = 20\ \mathrm{mm}$, $\sigma_r = 0$

$$0 = A_2 - \frac{B_2}{400}$$

$$B_2 = 400A_2$$

At $r = 10\ \mathrm{mm}$, $\sigma_r = -p$ (interface pressure)

$$-p = A_2 - \frac{B_2}{100}$$

Substitute to remove B_2 from the expression:

$$-p = A_2\left(1 - \frac{400}{100}\right)$$

And rearrange to give:

$$A_2 = \frac{1}{3}p$$

[2 marks]

Therefore:

$$B_2 = 400 \times \frac{1}{3}p$$

$$B_2 = \frac{400}{3}p$$

[2 marks]

The general expressions for the inner cylinder (1) are:

$$\sigma_{r1} = -\frac{4}{3}\left(1 - \frac{25}{r^2}\right)$$

$$\sigma_{\theta 1} = -\frac{4}{3}p\left(1 + \frac{25}{r^2}\right)$$

And the general expressions for the outer cylinder (2) are:

$$\sigma_{r2} = \frac{1}{3}p\left(1 - \frac{400}{r^2}\right)$$

$$\sigma_{\theta 2} = \frac{1}{3}p\left(1 + \frac{400}{r^2}\right)$$

However, at this stage *p* is still unknown.

Now need to consider compatibility:

$$i_1 + i_2 = i = 0.01 \text{ mm}$$

[1 mark]

Recalling:

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}\left(\sigma_\theta - \nu(\sigma_r + \sigma_z)\right)$$

as $\sigma_z = 0$, this reduces to:

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

[1 mark]

At the outside of cylinder 1, $r = 10$ mm,

$$\frac{-i_1}{10} = \frac{1}{100000}(\sigma_\theta - \nu\sigma_r) = \frac{1}{100000}\left(-\frac{4}{3}p\right)\left(1 + \frac{25}{100} - 0.3\left(1 - \frac{25}{100}\right)\right)$$

$$i_1 = 1.37 \times 10^{-4} p$$

[1 mark]

At the inside of cylinder 2, $r = 10$ mm,

$$\frac{+i_2}{10} = \frac{1}{210000}(\sigma_\theta - \nu\sigma_r) = \frac{1}{210000}\left(\frac{1}{3}p\right)\left(1 + \frac{400}{100} - 0.3\left(1 - \frac{400}{100}\right)\right)$$

$$i_2 = 9.37 \times 10^{-5} p$$

[1 mark]

As:

$$i_1 + i_2 = i = 0.01 \text{ mm}$$

$$1.37 \times 10^{-4} p + 9.37 \times 10^{-5} p = 0.01$$

$$2.307 \times 10^{-4} p = 0.01$$

[1 mark]

we can now rearrange to determine a value for *p:*

$$p = \frac{0.01}{2.307 \times 10^{-4}} = \mathbf{43.3\ MPa}$$

[2 marks]

22.

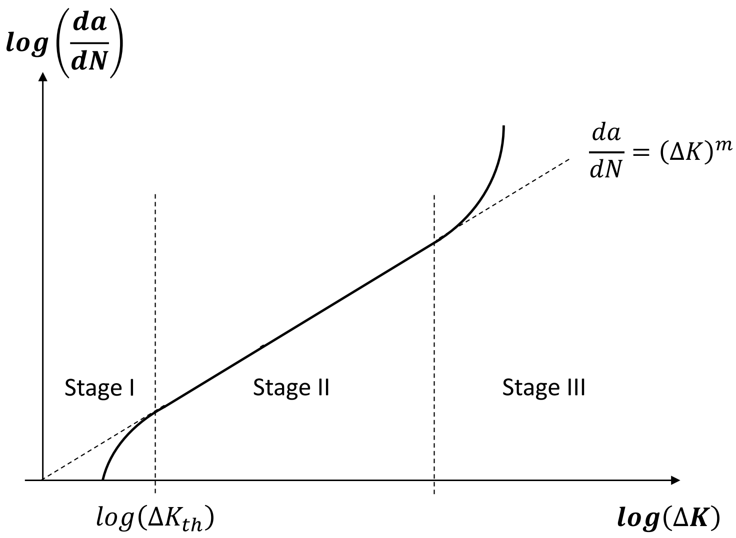
(a)

Paris showed that crack growth can be represented by the following empirical relationship:

$$\frac{da}{dN} = C(\Delta K)^m$$

Where C and m are empirically determined material constants. There are 3 stages in the relationship between Crack growth Rate and Stress Intensity Factor, as shown in the following figure.

[1 mark]



[2 marks]

<u>Stage I:</u> Below ΔK_{th}, no observable crack growth occurs.

<u>Stage II:</u> This region shows an essentially linear relationship between Crack growth Rate and Stress Intensity Factor (on a log-log scale), where m is the slope and C is the vertical axis intercept.

<u>Stage III:</u> Rapid crack growth occurs and little life is involved.

[1 mark]

(b)

Stress Intensity Factor is given as:

$$K_I = Y\sigma\sqrt{\pi a}$$

[2 marks]

where the geometry (and therefore boundaries) affect the value of Y. For example, for a for a crack in an infinite plate, Y = 1 and for small values of a/W, Y = 1.12 (where W is the width of the plate).

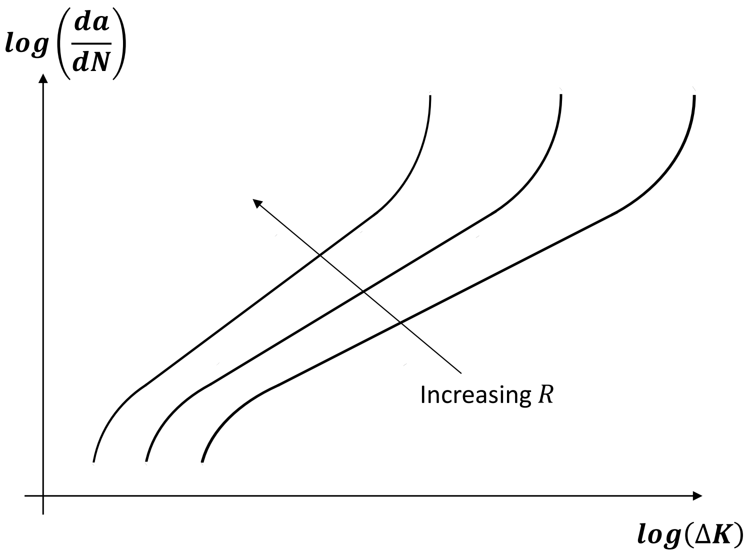
[1 mark]

Similarly for K_{II} and K_{III}. Where Y is a function of the crack and component (geometry).

[1 mark]

(c)

As mean stress (and therefore R) is increased, fatigue life is decreased as shown in the following figure:



[4 mark]

(d)

$$K_I = 1.12\sigma\sqrt{\pi a}$$

$$\therefore K_{I_{cr}} = 1.12 \times \frac{3}{5}\sigma_y\sqrt{\pi a_{cr}} \qquad (1)$$

[3 marks]

where

$$K_{I_{cr}} = 175MPa\sqrt{m}$$

and

$$\sigma_y = 210MPa$$

Substituting these values for $K_{I_{cr}}$ and σ_y into (1) gives:

$$175 = 1.12 \times \frac{3}{5} \times 210 \times \sqrt{\pi a_{cr}}$$

[2 marks]

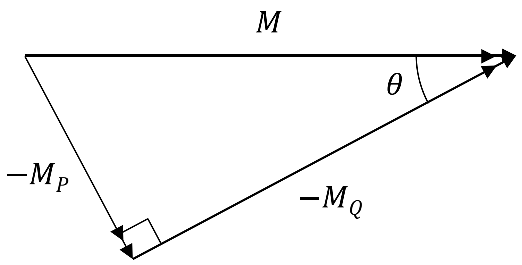
$$\therefore a_{cr} = \left(\frac{175}{1.12 \times \frac{3}{5} \times 210} \right)^2 \times \frac{1}{\pi} = 0.489m = \boldsymbol{489mm}$$

[3 marks]

23.

(a) Position/orientation of the neutral axis

Resolving M onto principal axes:



Note: The minus signs for $-M_P$ and $-M_Q$ are due to the components of M on the P and Q axes being in the negative direction.

[2 marks]

From the triangle above:

$$sin\theta = \frac{-M_P}{M}$$

$$\therefore M_P = -Msin\theta$$

and,

$$cos\theta = \frac{-M_Q}{M}$$

$$\therefore M_Q = -Mcos\theta$$

[1 mark]

At the neutral axis:

$$\sigma_b \left(= \frac{M_P Q}{I_P} - \frac{M_Q P}{I_Q} \right) = 0$$

[1 mark]

$$\therefore \frac{Q}{P} = \frac{M_Q I_P}{M_P I_Q}$$

$\frac{Q}{P}$ represents the gradient of the neutral axis on the $P-Q$ axes. Therefore, the angle of the neutral axis with respect to the $P-Q$ axes, α, is:

$$\alpha = \tan^{-1}\left(\frac{Q}{P}\right) = \tan^{-1}\left(\frac{M_Q I_P}{M_P I_Q}\right) = \tan^{-1}\left(\frac{-Mcos\theta \times I_P}{-Msin\theta \times I_Q}\right) = \tan^{-1}\left(\frac{I_P cos\theta}{I_Q sin\theta}\right)$$

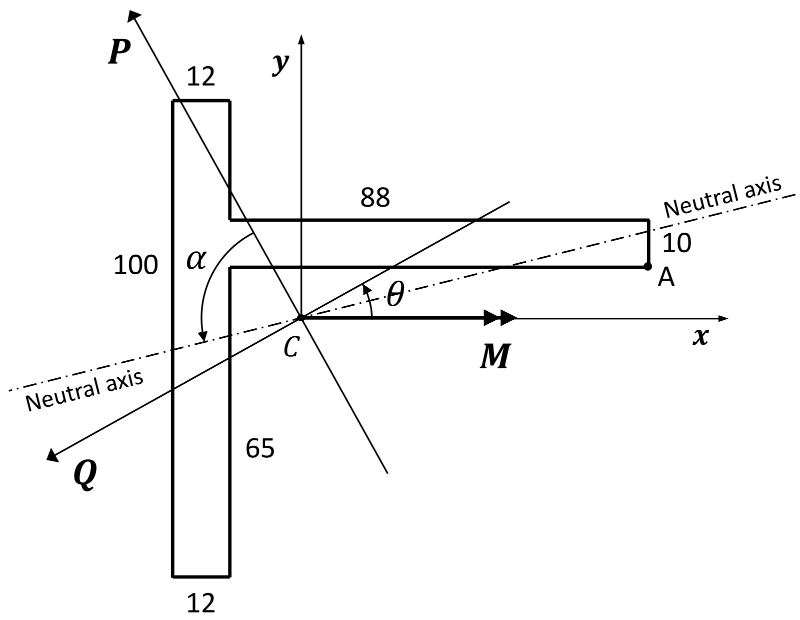
[2 marks]

Substituting in values for I_P, I_Q and θ gives:

$$\therefore \boldsymbol{\alpha = 76.46°}$$

[1 mark]

Therefore, the neutral axis drawn on top of the cross section is as follows:



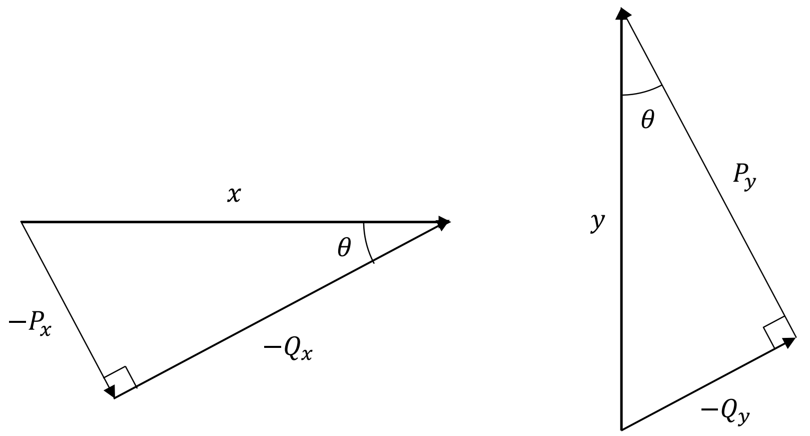
(b) Bending stress at position A

The bending stress at position A can be calculated as:

$$\sigma_b{}^A = \frac{M_P Q_A}{I_P} - \frac{M_Q P_A}{I_Q} \qquad (1)$$

[2 marks]

where P_A and Q_A are the co-ordinates of position A on the $P-Q$ axes. In order to find these $P-Q$ co-ordinates, a translation from $x-y$ to $P-Q$ must be determined as follows:



$$sin\theta = \frac{-P_x}{x}$$

$$\therefore P_x = -xsin\theta$$

$$cos\theta = \frac{P_y}{y}$$

$$\therefore P_y = ycos\theta$$

$$cos\theta = \frac{-Q_x}{x}$$

$$\therefore Q_x = -xcos\theta$$

$$sin\theta = \frac{-Q_y}{y}$$

$$\therefore Q_y = -ysin\theta$$

$$P = P_x + P_y$$

$$\therefore P = -xsin\theta + ycos\theta$$

$$Q = Q_x + Q_y$$

$$\therefore Q = -xcos\theta - ysin\theta$$

Note: The minus signs for $-P_x$, $-Q_x$ and $-Q_y$, are due to the components of x on the P and Q axes and the component of y on the Q axis being in the negative direction.

[2 marks]

Using these translations to determine the $P - Q$ co-ordinates:

At A, $x_{\mathrm{A}} = 72.85$ and $y_A = 6.54$,

$$\therefore P_{\mathrm{A}} = -x_{\mathrm{A}}sin\theta + y_{\mathrm{A}}cos\theta$$

$$\therefore Q_{\mathrm{A}} = -x_{\mathrm{A}}cos\theta - y_{\mathrm{A}}sin\theta$$

[1 mark]

Substituting in values for x_{A}, y_{A} and θ gives:

At A, $P_{\mathrm{A}} = -29.45$ and $Q_{\mathrm{A}} = -66.95$

[1 mark]

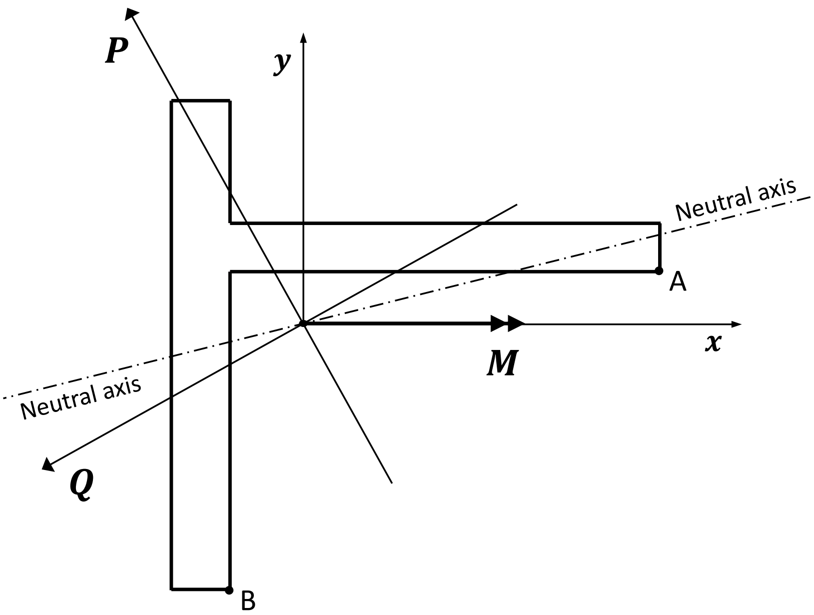
Substituting values for M_P, M_Q, I_P, I_Q, P_{A} and Q_{A} into equation (1):

$$\boldsymbol{\sigma_b}^{\mathbf{A}} = \mathbf{-25.087\ MPa}\ (\boldsymbol{compressive})$$

[1 mark]

(c) Location and value of the maximum compressive bending stress

Maximum compressive bending stress is assumed to occur at position B (same side of the neutral axis as position A, as $\sigma_b{}^{\mathrm{A}}$ is also compressive) as this appears to be the furthest point from the neutral axis.



[1 mark]

At B, $x_{\mathrm{B}} = -15.15$ and $y_{\mathrm{B}} = -58.46$,

[1 mark]

$$P_{\mathrm{B}} = -x_{\mathrm{B}} sin\theta + y_{\mathrm{B}} cos\theta$$

and,

$$Q_{\mathrm{B}} = -x_{\mathrm{B}} cos\theta - y_{\mathrm{B}} sin\theta$$

Substituting in values for x_{B}, y_{B} and θ gives:

At B, $P_{\mathrm{B}} = -43.88$ and $Q_{\mathrm{B}} = -14.96$

[2 marks]

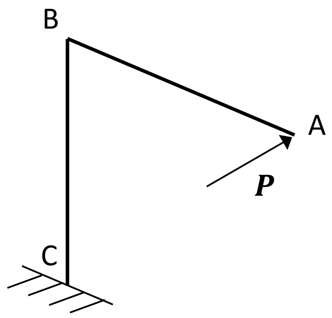
Substituting values for M_P, M_Q, I_P, I_Q, P_{B} and Q_{B} into equation (1):

$$\boldsymbol{\sigma_b}^{\boldsymbol{B}} = -\mathbf{75.812\ MPa}\ (\boldsymbol{compressive})$$

[2 marks]

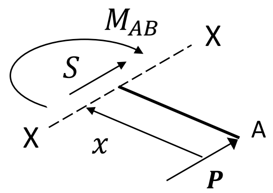
24.

Labelling the sections of the beam AB and BC:



Section AB (*bending only*)

Sectioning the beam along the length AB and drawing a free body diagram (FBD):



[2 marks]

Taking moments about X-X:

$$M_{\mathrm{AB}} = Px$$

[1 mark]

Substituting this into the equation for Strain Energy in a beam under bending gives,

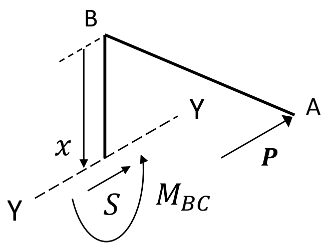
$$U_{\mathrm{AB}} = \int_0^L \frac{{M_{\mathrm{AB}}}^2}{2EI} dx = \frac{P^2}{2EI} \int_0^L x^2 \, dx = \frac{P^2}{2EI} \left[\frac{x^3}{3} \right]_0^L$$

$$\therefore U_{\mathrm{AB}} = \frac{P^2 L^3}{6EI}$$

[2 marks]

Section BC (*bending & torsion*)

Sectioning the beam along the length BC and drawing an FBD:



[2 marks]

Taking moments about Y-Y:

$$M_{\mathrm{BC}} = Px$$

and,

$$T_{\mathrm{BC}} = PL$$

[2 marks]

Substituting these into the equations for Strain Energy in a beam under bending and in a beam under torsion, respectively, gives:

$$U_{\mathrm{BC}} = \int_0^L \frac{{M_{\mathrm{BC}}}^2}{2EI} dx + \int_0^L \frac{{T_{\mathrm{BC}}}^2}{2GJ} dx = \frac{P^2}{2EI}\int_0^L x^2\, dx + \frac{P^2L^2}{2EI}\int_0^L dx = \frac{P^2}{2EI}\left[\frac{x^3}{3}\right]_0^L + \frac{P^2L^2}{2GJ}[x]_0^L$$

$$\therefore U_{\mathrm{BC}} = \frac{P^2L^3}{6EI} + \frac{P^2L^3}{2GJ}$$

[2 marks]

Total Strain Energy

$$U = U_{\mathrm{AB}} + U_{\mathrm{BC}} = \frac{P^2L^3}{3EI} + \frac{P^2L^3}{2GJ} = \frac{P^2L^3}{2}\left(\frac{2}{3EI} + \frac{1}{GJ}\right)$$

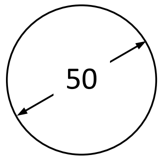
[1 mark]

Deflection Calculation using Castigliano's Theorem

$$u = \frac{\delta U}{\delta P} = PL^3\left(\frac{2}{3EI} + \frac{1}{GJ}\right) \qquad (1)$$

[2 marks]

For a circular cross-section, as shown in the diagram below, the 2nd moment of area and polar 2nd moment of area, I and J, respectively, are calculated as follows:



$$I = \frac{\pi D^4}{64} = \frac{\pi \times 50^4}{64} = 306{,}796.16 \text{ mm}^4$$

and,

$$J = \frac{\pi D^4}{32} (= 2I) = 613{,}591.32 \text{ mm}^4$$

[2 marks]

Substituting values for P, L, E, I, G and J into equation (1) gives:

$$u = 82.58 \text{ mm}$$

[2 marks]

This deflection is greater than the initial gap between beam tip and the wall (i.e. $82.58 \text{ mm} > 75 \text{ mm}$), therefore **the tip of the beam does touch the wall**.

[2 marks]

25.

(a)

$$A = 0.02 \times 0.03 = 6 \times 10^{-4}\ \text{m}^2$$

$$L = 0.6\ \text{m}$$

$$E = 70 \times 10^{9}\ \text{Pa}$$

$$\frac{AE}{L} = \frac{6 \times 10^{-4}\ \times 70 \times 10^{9}}{0.6} = 70 \times 10^{6}\ \text{N/m}$$

[2 marks]

The stiffness matrix of a truss element is:

$$[K_e] = \left(\frac{AE}{L}\right)\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Element 1, angle $\theta = 75°$, $cos\ \theta = 0.259$ and $sin\ \theta = 0.966$

$$\therefore [K_{e1}] = (70 \times 10^{6})\begin{bmatrix} 0.067 & 0.250 & -0.067 & -0.250 \\ 0.250 & 0.933 & -0.250 & -0.933 \\ -0.067 & -0.250 & 0.067 & 0.250 \\ -0.250 & -0.933 & 0.250 & 0.933 \end{bmatrix}$$

[3 marks]

Element 2, angle $\theta = 180°$, $cos\ \theta = -1$, $sin\ \theta = 0$

$$[K_{e2}] = (70 \times 10^{6})\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[3 marks]

Overall stiffness matrix for structure

$$[\boldsymbol{K}] = \left(\mathbf{70 \times 10^{6}}\right)\begin{bmatrix} \mathbf{0.067} & \mathbf{0.250} & \mathbf{-0.067} & \mathbf{-0.250} & \mathbf{0} & \mathbf{0} \\ \mathbf{0.250} & \mathbf{0.933} & \mathbf{-0.250} & \mathbf{-0.933} & \mathbf{0} & \mathbf{0} \\ \mathbf{-0.067} & \mathbf{-0.250} & \mathbf{1.067} & \mathbf{0.250} & \mathbf{-1} & \mathbf{-0} \\ \mathbf{-0.250} & \mathbf{-0.250} & \mathbf{0.250} & \mathbf{0.933} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

[4 marks]

(b)

Horizontal and vertical components of force at B:

$$F_{BH}\ (F3) = 20{,}000cos(-15) = 19318\ \mathrm{N}$$

$$F_{BV}\ (F4) = 20{,}000sin(-15) = -5176\ \mathrm{N}$$

[2 marks]

Overall equations:

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = (70 \times 10^6) \begin{bmatrix} 0.067 & 0.250 & -0.067 & -0.250 & 0 & 0 \\ 0.250 & 0.933 & -0.250 & -0.933 & 0 & 0 \\ -0.067 & -0.250 & 1.067 & 0.250 & -0.500 & -0.500 \\ -0.250 & -0.250 & 0.250 & 0.933 & -0.500 & -0.500 \\ 0 & 0 & -0.500 & -0.500 & 0.500 & 0.500 \\ 0 & 0 & -0.500 & -0.500 & 0.500 & 0.500 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

Applying BCs, $u_1 = u_2 = u_5 = u_6 = 0$, reduces the problem to:

$$\begin{bmatrix} F_3 \\ F_4 \end{bmatrix} = (70 \times 10^6) \begin{bmatrix} 1.067 & 0.250 \\ 0.250 & 0.933 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

Applying forces

$$\begin{bmatrix} 19318 \\ -5176 \end{bmatrix} = (70 \times 10^6) \begin{bmatrix} 1.067 & 0.250 \\ 0.250 & 0.933 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

[2 marks]

therefore:

$$19318 = (7.47 \times 10^7)u_3 + (1.75 \times 10^7)u_4 \qquad (1)$$

$$-5176 = (1.75 \times 10^7)u_3 + (6.53 \times 10^7)u_4 \qquad (2)$$

from (1):

$$u_3 = \frac{19318 - (1.75 \times 10^7)u_4}{7.47 \times 10^7}$$

$$\therefore u_3 = 2.586 \times 10^{-4} - 0.23u_4$$

subs this into (2):

$$-5176 = 1.75 \times 10^7 (2.586 \times 10^{-4} - 0.23u_4) + (6.53 \times 10^7)u_4$$

$$\therefore -5176 = 4.526 \times 10^3 + (6.1275 \times 10^7)u_4$$

$$\therefore u_4 = \frac{-5176 - 4.526 \times 10^3}{6.1275 \times 10^7}$$

$$\therefore \boldsymbol{u_4 = -1.584 \times 10^{-4}\ \mathrm{m}}$$

[2 marks]

subs this into (1)

$$u_3 = \frac{19318 - 1.75 \times 10^7 \times -1.584 \times 10^{-4}}{7.47 \times 10^7}$$

$$\therefore \boldsymbol{u_3 = 2.957 \times 10^{-4}\ \mathrm{m}}$$

[2 marks]